Fixed Point Algorithm based on Proximity and Precondition Operator for High-Resolution Image Reconstruction with Displacement-Errors

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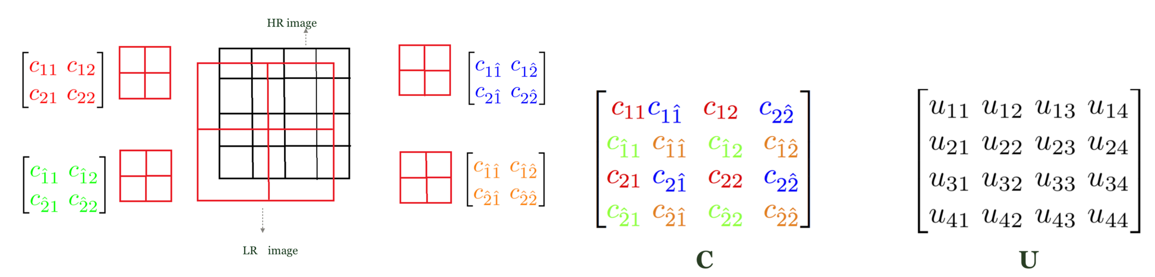
Introduction

High resolution image reconstruction arises in many applications, such as remote sensing, surveillance, and medical imaging. It refers to the reconstruction of high-resolution image from multiple low-resolution, shifted, degraded samples of a true image. In this work we consider Bose-Boo observation model to reconstruct HR image. Displacement error is inevitable during the capture of low resolution images. The goal for this work is to address the specific issue of high resolution image reconstruction with displacement error by proposing a novel optimization model invoking the Moreau envelop, denoted by $\mathrm{env}_{\ell^1}/\mathrm{TV}$, which has been studied and explored extensively in the present work, to compensate the bad performance of L_2/TV deblurring model while dealing with realistic noisy data and a fixed point algorithm with precondition is proposed, whose convergence condition has been given in the meantime, and an adaptive strategy has been given for better PSNR appearance and faster convergence rate. Numerical results shows the method of $\mathrm{env}_{\ell^1}/\mathrm{TV}$ model combine new algorithm has very fast convergent speed in dealing with both Gaussian noise and salt $\&$ pepper noise.

Bose-Boo Observation Model[2]



(a) (b)

Figure 1: Bose-Boo model for amplifying four times. (a) shows where LR sensors will be placed, while (b) shows the composite matrices of LR images and high resolution image.

The relationship of HR image and LR images can be formed as

$$\vec{C} = A_L(\varepsilon^x, \varepsilon^y)\vec{U} + \eta$$

where

$$A_L(\varepsilon^x, \varepsilon^y) = \frac{1}{L \times L}\sum_{p,q=1}^{L} D_{p,q} A_L(\varepsilon^x_{p,q}, \varepsilon^y_{p,q}),$$

in which

$$D_{p,q} := J_q \otimes J_p,\ A_L(\varepsilon^x_{p,q}, \varepsilon^y_{p,q}) := A_L(\varepsilon^y_{p,q}) \otimes A_L(\varepsilon^x_{p,q}),$$

$$A_L(\varepsilon) = \frac{1}{L} Toeplitz(a,b) + \frac{1}{L} Hankel(c,d)$$

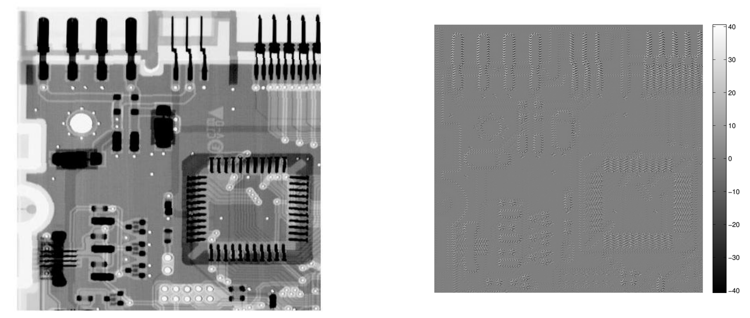
where

$$a = [\underbrace{1,\dots,1}_{\frac{L}{2}}, \frac{1}{2}+\varepsilon, 0, \dots, 0], \quad b = [\underbrace{1,\dots,1}_{\frac{L}{2}}, \frac{1}{2}-\varepsilon, 0, \dots, 0]$$

$$c = [\underbrace{1,\dots,1}_{\frac{L}{2}-1}, \frac{1}{2}+\varepsilon, 0, \dots, 0], \quad d = [\underbrace{1,\dots,1}_{\frac{L}{2}-1}, \frac{1}{2}-\varepsilon, 0, \dots, 0],$$

η denotes the addictive noise.

Optimization Model



(a) (b)

Figure 2: Displacement error. (a) Original image, (b) random displacement error with $\varepsilon_{\max} = 0.99$.

Definition 0.1. *Let $f : \mathcal{H} \to (-\infty, +\infty]$ and let $\tau \in \mathbb{R}_{++}$. The Moreau envelop [2] of f respects to parameter τ is*

$$\mathrm{env}_{\tau f}(x) = \min_{y \in \mathcal{H}} \{f(y) + \frac{1}{2\tau}\|x-y\|_2^2\}. \tag{1}$$

.

Proposition 0.2. *The moreau envelope of f satisfies that $\mathrm{env}_{\tau f}(x)$ is continuous differentiable and the gradient of it is*

$$\nabla \mathrm{env}_{\tau f}(x) = \frac{1}{\tau}(I - \mathrm{prox}_{\tau f}(x)).$$

We can prove that $\mathrm{env}_{\tau\|\ \|_1}(x)$ has β-Lipschitz continuous gradient. Combine all these analyses, we propose a modified model (2):

$$u^* = \arg\min_{u \in \mathbb{R}^n} \{\mu\|u\|_{\mathrm{TV}} + \mathrm{env}_{\tau\|\ \|_1}(Au - c), c \in \mathbb{R}^n\}. \tag{2}$$

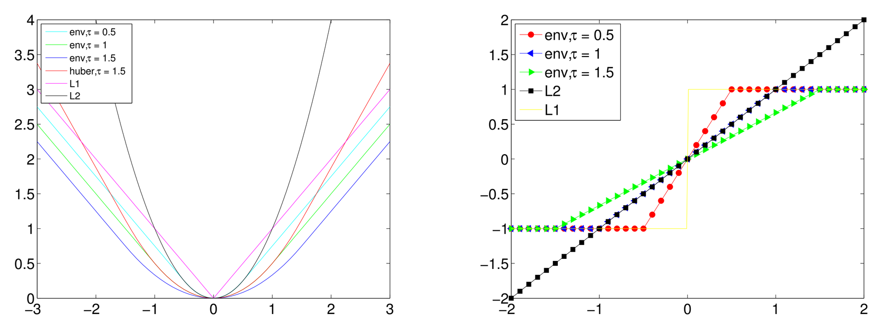


Figure 3: $\mathbb{R}^1$ situation. (a) The figures of L1 norm, L2 nom and Moreau envelop. (b) The figures of influence functions of L1 norm, L2 nom and Moreau envelop.

FP^3 Algorithm

For optimization model

$$u^* = \arg\min_{u \in \mathcal{H}} \{p_1(Bu) + p_2(Au), u \in \mathbb{R}^n\} \tag{3}$$

where $p_1, p_2 \in s_0(\mathbb{R}^n)$, and $p_2(A\cdot)$ is differentiable with a β-Lipschitz continuous gradient. We have the following theorem:

Theorem 0.3. *Let $p_1 \in s_0(\mathbb{R}^m)$, $p_2 \in s_0(\mathbb{R}^n)$, $\lambda > 0$, $A \in \mathbb{R}^{m_1 \times n}$, $B \in \mathbb{R}^{m_2 \times n}$. If u is a solution of model (2), as function p_2 is differentiable, then for any $\lambda > 0$ and $S \in \mathbb{S}^n_+$, there exists a vector $v \in \mathbb{R}^{m_2}$, such that the pair $(v, u) \in \mathbb{R}^{m_2} \times \mathbb{R}^n$ satisfies the following coupled fixed point equations:*

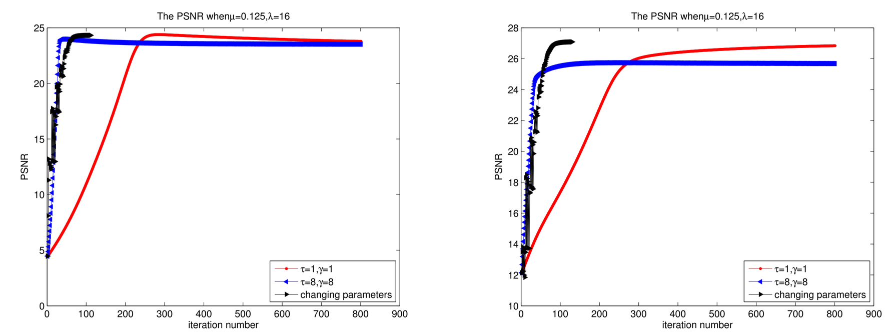
$$v = (I - \mathrm{Prox}_{\lambda p_1})(v + Bu) \tag{4}$$

$$u = u - S(A^T \nabla p_2(Au) + \frac{1}{\lambda} B^T v). \tag{5}$$

Conversely if $u \in \mathbb{R}^n$ satisfies equations (4)-(5) for some $\lambda > 0$ and $S \in \mathbb{S}^n_+$, then it is a solution of model (3).

Algorithm 1. *Fixed point algorithm based on proximity and precondition operator,* FP^3.
Step 1: set S being a symmetric positive definite matrix and $\|S\|_2^2 < \frac{1}{2\beta}$,
Step 2: for $k = 0, 1, 2, \cdots$,
$v_{k+1} = (I - \mathrm{Prox}_{\lambda p_1})(Bu_k + v_k)$,
$u_{k+1} = u_k - S(A^T \nabla p_2(Au_k) + \frac{1}{\lambda} B^T(2v_{k+1} - v_k))$.
end for

Algorithm 2. *Fixed point algorithm based on proximity operator accelerated by adaptive parameters*
Step 1: Set $u_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^m$, $Adaptive\ Frequency$, S being a symmetric positive definite matrix and $\|S\|_2^2 < \frac{1}{2\beta}$,
Step 2: For $k = 0, 1, 2, \dots$
$z = rem(m1, Adaptive\ Frequency)$;
if $z == 0$ $\quad i = i + 1;$ *else* *end*
$\tau = 128/2^i;$ $\quad s = \tau;$
$v_{k+1} = (I - \mathrm{Prox}_{\lambda p_1})(Bu_k + v_k)$,
$u_{k+1} = u_k - S(A^T \nabla p_2(Au_k) + \frac{1}{\lambda} B^T(2v_{k+1} - v_k))$.
end for.



(a) (b)

Figure 4: The PSNR of reconstructioned images with different paramete τ and s. (a) are the reconstruction results of 'electricpad', (b) are the reconstruction results of 'phantom'.

Numerical Results

			$\lambda = 1$	$\lambda = 2$	$\lambda = 4$	$\lambda = 8$	$\lambda = 16$
$\mathrm{TOL} = 10^{-3}$	$\delta^2 = 10^{-3}$	L_2/TV	32.4717(40)	**32.5323(3)**	**32.5888(3)**	**32.6400(3)**	32.7007(4)
		$\mathrm{env}_{\ell^1}/\mathrm{TV}$	32.5149(70)	32.5167(70)	**32.5582(61)**	**32.5629(61)**	32.6356(52)
	$E_1 = 10^{-3}$	L_2/TV	32.6833(40)	**32.7272(2)**	**32.7743(2)**	**32.8232(2)**	**32.8719(2)**
		$\mathrm{env}_{\ell^1}/\mathrm{TV}$	34.7588(70)	34.7594(70)	34.6546(61)	34.6483(61)	**34.5120(52)**
$\mathrm{TOL} = 10^{-4}$	$\delta^2 = 10^{-3}$	L_2/TV	32.8640(100)	32.8666(87)	**32.9078(31)**	32.9405(41)	32.8904(55)
		$\mathrm{env}_{\ell^1}/\mathrm{TV}$	32.4752(91)	32.4841(82)	**32.4835(80)**	**32.4926(80)**	**32.5133(80)**
	$E_1 = 10^{-3}$	L_2/TV	33.1906(100)	33.2609(76)	**33.3610(28)**	33.5251(35)	33.7351(46)
		$\mathrm{env}_{\ell^1}/\mathrm{TV}$	34.8514(91)	34.8317(87)	**34.8250(86)**	**34.8472(86)**	**34.8778(86)**

Table 1: PSNR(dB) results of reconstructed image 'lena' with different λ under different stopping criterion.

Figure 5: Reconstruction results of different level of salt & pepper noises by using two method. The first column are the polluted images with Gaussian noise of $\delta^2 = 1.0 \times 10^{-4}, 1.0 \times 10^{-3}, 1.0 \times 10^{-3}$. The second column are reconstructed images by using L_2/TV model and Algorithm 1, while the third column are reconstructed images by using $\mathrm{env}_{\ell^1}/\mathrm{TV}$ model and Algorithm 2.

Figure 6: Reconstruction results of different level of salt & pepper noises by using two method. The first column are the polluted images with $0.01\%, 0.1\%, 1\%$ salt $\&$ pepper noise. The second column are reconstructed images by using L_2/TV model and Algorithm 1, while the third column are reconstructed images by using $\mathrm{env}_{\ell^1}/\mathrm{TV}$ model and Algorithm 2.

Conclusions

1. Propose a new optimization model based on the property of Bose-Boo high resolution reconstruction model with displacement errors.
2. Develop an new algorithm for deblurring problem with the fidelity term having β-Lipschitz continuous gradient.
3. Prove the convergence of the new algorithm for L_2/TV model and $\mathrm{env}_{\ell^1}/\mathrm{TV}$ model in a new way which is easy to achieve.
4. Prove the stability and effectiveness by doing several numerical experiments.

References

[1] Heinz H Bauschke and Patrick L Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. Springer, 2011.

[2] NK Bose and KJ Boo. High-resolution image reconstruction with multisensors. *International Journal of Imaging Systems and Technology*, 9(4):294–304, 1998.