Semidefinite Programming, Combinatorial Optimization and Real Algebraic Geometry

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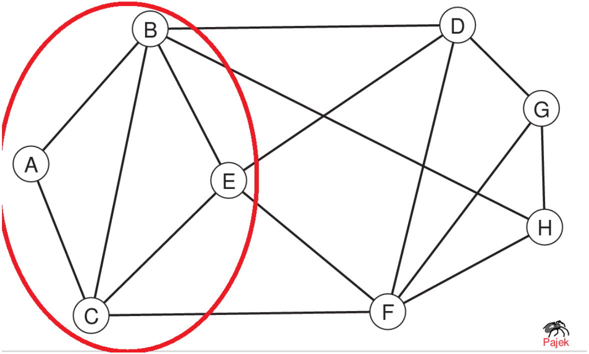
Outline

Some notation

- I - unit matrix,
- $X \in \mathbb{R}^{m \times n} \Rightarrow x = \mathrm{vec}(X) \in \mathbb{R}^{mn}$
- $\langle x, y \rangle = x^T y = \sum_i x_i y_i$
- $\langle A, B \rangle = \mathrm{tr}(A^T B) = \sum_{i,j} a_{ij} b_{ij}$
- $\mathbf{e}$ - vector of all ones.

The Max Cut problem

- Given weighted graph $G = (V, E)$ with edge weights W:



The max cut problem (MCP)

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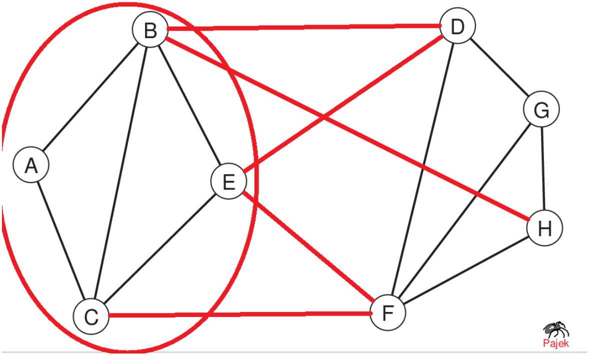
- Given weighted graph $G = (V, E)$ with edge weights W.
- Find $S \subset V$ with maximum cut edges.



The max cut problem (MCP)

MCP - formally

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Def. MCP

Given weighted graph $G = (V, E)$ with edge weights W, **find** $S \subset V$ such that

$$\mathrm{cut}(S) := \sum_{i \in S, j \in V \setminus S} w_{ij}$$

is maximum.

MCP - int. prog. formulation

Def. MCP

Given weighted graph $G = (V, E)$ with edge weights W, solve

$$\begin{aligned} \max \quad & \tfrac{1}{4}\textstyle\sum_{i,j} w_{ij}(1 - x_i x_j) \\ \mathbf{x} \; & \in \; \{-1, 1\}^n \end{aligned}$$

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Solution for MCP: $S = \{i \colon x_i = 1\}$.

Thm.: (Karp, 1972) MCP is an NP-complete problem.

Thm.: MCP is polynomial if graph is **planar** (Orlova, Dorfman), **weakly** bipartite (Grötchel, Pulleyblank), graphs **without long odd cycles** (Grötchel and Nemhauser), **line** graphs (Arbib), graphs with **bounded tree width** (Bodlaender, Jansen) and some others.

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Def. MCP-SDP

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Idea: Introduce $Y = [y_{ij}], \quad y_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$.

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Back: Compute $v_i \in \mathbb{R}^n$ such that $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = y_{ij}$.

How to obtain a **good** cut?

1. **Input:** $\{\mathbf{v}_i\}$ obtained from SDP relaxation MCP-SDP.
2. Generate random $\mathbf{r} \in S_{n-1}$.
3. **Define** $S = \{i \colon \langle r, v_i \rangle \geq 0\}$.

Thm.: (Goemans, Williamson, 1995) Let $\mathbf{r} \in S_{n-1}$ be obtained by uniform distribution yielding S. Then

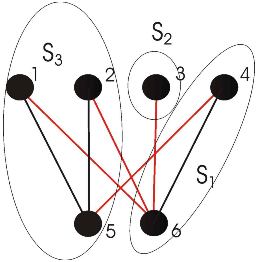
$$\mathbb{E}(\mathrm{cut}(S)) > 0.87856 \cdot \mathrm{cut}(S^{opt}).$$

Solving to optimality

- Common approach: Branch and Bound.
- Good lower bound and upper bounds are needed.
- Good upper bounds obtained by SDP relaxations (improved by further constraints - see Rendl, Rinaldi, Wiegele 2010).
- Biq Mac (Wiegele) - web based solver for MCP (up to 300 vertices).
- Solving MCP and and MkCP in practise: heuristics.

The graph partitioning problem - a picture

- Partition the nodes of a graph into sets with **prescribed** sizes such that the number of edges between different sets is minimal.



Slika: $\text{cut}(S_1, S_2, S_3) = 4$

Motivations and complexity

- GP problem appears in floor planning, analysis of networks etc.
- GP is connected with vertex separator problem and bandwidth problem.

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- GP is connected with vertex separator problem and bandwidth problem.
- If $k = 2$, $m_1 = \lceil \frac{n}{2} \rceil$, $m_2 = \lfloor \frac{n}{2} \rfloor$, we get an NP-complete graph bisection problem as a special case.

Application to the graph partitioning problem

- **INPUT** Graph $G = (V, E)$, $k \in \mathbb{N}$, $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k$.

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$$\boxed{\begin{array}{rcl} OPT_{GPP} & = & \min \quad \frac{1}{2}\langle X,\, AXB\rangle \\ \text{s. t.} \quad X & \in & \mathbb{R}_+^{n\times k}, \\ X^T X & = & M := \mathrm{Diag}(\mathbf{m}) \\ \mathrm{diag}(XX^T) & = & u_n \end{array}}$$

A...adjacency matrix of G, $B = J - I$.

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Adding redundant constraints

Every partition matrix X satisfies $XM^{-1}X^T \preceq I$, $\mathbf{e}_n^T X \mathbf{e}_k = n$.

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We use $\mathbf{x} = \mathrm{vec}(X)$ and $\langle X, AXB\rangle = \langle B \otimes A,\ \mathbf{x}\mathbf{x}^T\rangle$.

SDP bounds for GPP

- We introduce $V = \mathbf{x}\mathbf{x}^T \in \mathcal{S}_{kn}^+$.

$$
\begin{aligned}
OPT_{GPP} \quad \geq \quad OPT_{DH} \quad = \quad & \min \ \langle B \otimes A, V \rangle \\
& V \in \mathcal{S}_{kn}^+, \ W \ \in \ \mathcal{S}_n^+ \\
& \textstyle\sum_{i=1}^k \frac{1}{m_i} V^{ii} + W \ = \ I, \ \ \langle I, V^{ij} \rangle \ = \ m_i \delta_{ij}, \ \forall i,j \\
& \langle I \otimes E_{ii}, \ V \rangle = 1, \ 1 \leq i \leq n
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& \langle I \otimes E_{ii},\ V \rangle = 1,\ 1 \leq i \leq n
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$$

Thm. (P. 2009) The semidefinite bound OPT_{DH} is exactly the **Donath-Hoffman** eigenvalue bound for OPT_{GPP}

$$
OPT_{DH} = \max \left\{ \frac{1}{2} \sum_{i=1}^{k} m_{k-i+1} \lambda_i(L+D) \colon D = \mathrm{Diag}(\mathbf{d}),\ \mathbf{e}^T \mathbf{d} = 0 \right\}
$$

Improved SDP bounds for GPP

- If we add $\mathbf{e}_n^T X \mathbf{e}_k = n$ (actually: $\langle X, J_n X J_k \rangle = n^2$), we obtain:

$$\begin{array}{rcll} OPT_{GPP} & \geq & OPT_{\text{new1}} & = \min \ \langle B \otimes A, V \rangle \\ & & V \in \mathcal{S}_{kn}^+, \ W \in \mathcal{S}_n^+ & \\ & & \sum_{i=1}^{k} \frac{1}{m_i} V^{ii} + W = I, \ \langle I, V^{ij} \rangle = m_i \delta_{ij}, \ \forall i,j & \\ & & \langle I \otimes E_{ii}, \ V \rangle = 1, \ 1 \leq i \leq n & \\ & & \langle J_{kn}, V \rangle = n^2. & \end{array}$$

Improved SDP bounds for GPP

- If we further add $x_{ij}x_{i\ell} = 0 \;\; \forall i, j, \ell, \; j \neq \ell,$, we obtain:

$$
\begin{array}{rclcl}
OPT_{GPP} & \geq & OPT_{\text{new2}} & = & \min \;\; \langle B \otimes A,\, V \rangle \\
& & V \in \mathcal{S}^+_{kn},\;\; W \;\in\; \mathcal{S}^+_n & & \\
& & \sum_{i=1}^k \frac{1}{m_i} V^{ii} + W \;=\; I,\;\; \langle I,\, V^{ij} \rangle \;=\; m_i \delta_{ij},\; \forall i, j & & \\
& & \langle I \otimes E_{ii},\;\; V \rangle = 1,\;\; 1 \leq i \leq n & & \\
& & \langle J_{kn}, V \rangle = n^2 & & \\
& & \langle E_{j\ell} \otimes E_{ii},\;\; V \rangle = 0 \;\; \forall i, j, \ell,\;\; j \neq \ell. & &
\end{array}
$$

Numerical results

name	n	$\lvert E\rvert$	OPT_{DH}	OPT_{new1}	OPT_{new2}	PRGP [WoZh 99]
g50.01	50	111	17.922	22.762	23.570	23.549
g50.02	50	256	81.956	95.920	99.983	99.423
g50.03	50	342	124.718	148.701	152.231	151.225
g50.04	50	478	204.303	236.697	242.578	242.063
g50.05	50	611	287.204	332.791	338.494	338.529
g50.06	50	759	378.250	440.780	443.184	442.966
g50.07	50	897	470.157	544.238	550.335	549.934
g50.08	50	984	530.486	615.035	620.326	620.168
g50.09	50	1098	618.867	719.456	722.990	722.270

Tabela: Semidefinite lower bounds for Graph partitioning problem, where $m = (5,\ 10,\ 15,\ 20)$

Stronger relaxations

- **Further strengthening:** adding new constraints, redundant from original constraints:
  - Triangle constraints for 0-1 programs.
  - Row sum/column sum constraints.
  - **Completely positive constraint:**

Def. X is **completely positive** iff $X = \sum_{i=1}^{r} \mathbf{x}_i \mathbf{x}_i^T$ for some $r \in \mathbb{N}$ and $\mathbf{x}_i \geq 0$.

Def. The cone of **completely positive** matrices $\mathcal{CP}$.

Def. The cone of **copositive** matrices $\mathcal{COP} = \{A \in \mathcal{S} : \mathbf{x}^T A \mathbf{x} \geq 0 \ \forall \mathbf{x} \geq 0\}$.

Second main result

Theorem 2 (P., 2009) Adding completely positive constraint $V \in \mathcal{CP}_{kn}$ to OPT_{new1} we obtain the **exact value** for GPP.

$$
\begin{aligned}
OPT_{GPP} \quad = \quad & \min \ \langle B \otimes A, \, V \rangle \\
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& \langle I \otimes E_{ii}, \ V \rangle = 1, \ 1 \leq i \leq n \\
& \langle J_{kn}, \ V \rangle = n^2.
\end{aligned}
$$

- **Proof technique:**
  - If V is feasible solution of completely positive program, then
    $$V = \sum_i \lambda_i \mathbf{x}_i \mathbf{x}_i^T,$$
    where $\mathbf{x}_i = \mathrm{vec}(X_i)$ and X_i feasible for GPP.
  - We explore the structure of the equations.

The contribution of copositive formulation

- The GPP problem remains NP-hard.
- GPP can be rewritten as completely positive program using other techniques (Burer 2008, P. 2007, P. 2009).
- We can approximate OPT_{GPP} using semidefinite approximations of cone $\mathcal{CP}_{kn}$ (De Klerk, Pasechnik, 2002) or direct heuristics (Bomze, Jarre, Rendl, 2009; Duer, Bundfuss, 2008, 2009).

Real algebraic geometry

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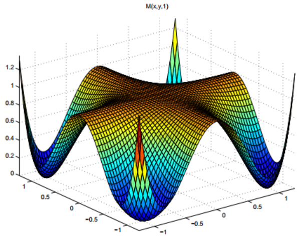
$$M(x,y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$



$$(x^2+y^2+1)\,M(x,y) = (x^2y-y)^2 + (xy^2-x)^2 + (x^2y^2-1)^2 + \\ + \frac{1}{4}(xy^3-x^3y)^2 + \frac{3}{4}(xy^3+x^3y-2xy)^2$$

Vir: http://www.mit.edu/ parrilo

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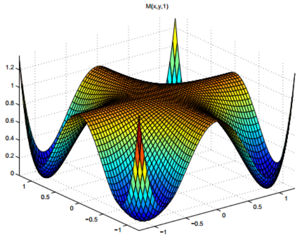

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Application of SDP in real algebraic geometry

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Ex. $f_A(\mathbf{x}) = \sum_{i,j} a_{ij} x_i^2 x_j^2 \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ **IFF** A is copositive.

Rem. The strong membership problems for $\mathcal{CP}$ and $\mathcal{COP}$ are NP-hard (Dickinson and Gijben, 2014).

- The Hilbert **17th problem** (1900): Is every non-negative polynomial with real coefficients a sum of squares of rational functions?
- Positive answer by Emil Artin in 1927.
- **Additional question:** Is every non-negative polynomial with real coefficients a sum of squares of real polynomials?
- The answer: **NO** (known already by Hilbert).

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Thm. $PSD_{n,2d} = SOS_{n,2d}$ iff

- $2d = 2$;
- $n = 1$;
- $n = 2, \ 2d = 4$.

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SOS polynomial - example

- Let $p(x_1, x_2) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$.
- p is SOS since

$$p(x_1, x_2) = \frac{1}{2}((2x_1^2 - 3x_2^2 + x_1 x_2)^2 + (x_2^2 + 3x_1 x_2)^2).$$

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- We can obtain sos decomp. by **Gram matrix method:**

Thm. Let $f \in \mathbb{R}[\mathbf{x}]$ with degree $2d$. f is SOS IFF there exists $Q \succeq 0$ such that

$$f(\mathbf{x}) = V_d^T Q V_d,$$

where V_d is the vector of all monomials of degree $\leq d$.

SOS polynomial - cnt.

- Let $p(x_1, x_2) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$.
- Corresponding SDP:

$$
\begin{aligned}
\min \quad & 0 \\
\text{p. p.} \quad & Q = \begin{bmatrix} 2 & a & 1 \\ a & 5 & 0 \\ 1 & 0 & b \end{bmatrix} \succeq 0, \\
& 2a + b = -1.
\end{aligned}
$$

where $V_2 = (x_1^2, x_2^2, x_1 x_2)^T$.

SOS polynomial - cnt.

If $b = 5$ we get

$$
\begin{aligned}
Q &= \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \\
&= \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}^T \cdot \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \succeq 0.
\end{aligned}
$$

Therefore:

$$
p(x_1, x_2) = \frac{1}{2}((2x_1^2 - 3x_2^2 + x_1x_2)^2 + (x_2^2 + 3x_1x_2)^2).
$$

Complexity of SOS SDP

- There are $\binom{n+d-1}{d}$ monomials of degree d. This is the order of Q.

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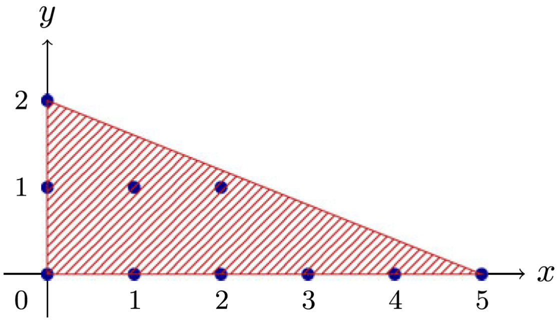
Thm. It is enough to consider only the monomials from one half of the **Newton polytope**:



Newton polytope for $f = 1 + 2y^2 - 4x^5$

Pólya Positivstellensatz

Thm. (Pólya, 1929, Hardy, Littlewood, Pólya, 1988, Powers, Reznick, 2001) Let $f \in \mathbb{R}[\mathbf{x}]$ be a homogeneous polynomial on $\mathbb{R}^n$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$. Then for some $r \in \mathbb{N}$, we have that all the coefficients of $(\mathbf{e}^T\mathbf{x})^r f(\mathbf{x})$ are non-negative (positive).

Application: In copositive programming - LP ali SDP certificates for copositivity: if $(\sum_i x_i^2)^r \sum_{i,j} A_{i,j} x_i^2 x_j^2$ has non-negative coefficients then A is copositive (LP problem).

Putinar Positivstellensatz

Thm. (Putinar, 1993) Let $m \in \mathbb{N}$ and $f, g_1 = 1, g_2, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$. If $f(\mathbf{x}) > 0$ for all

$$\mathbf{x} \in K := \{\mathbf{x} \in \mathbb{R}^n \colon g_i(\mathbf{x}) \geq 0, \text{ for } i = 1, \ldots, m\} \setminus \{\mathbf{0}\},$$

then there exists $s_1, \ldots, s_m \in SOS$ such that $f(\mathbf{x}) = \sum_{i=1}^m s_i g_i(\mathbf{x})$, provided e.g. K compact.

Application in optimization

- $f_{\text{inf}} = \inf\{f(\mathbf{x})\colon \mathbf{x} \in K\} = \sup\{\varepsilon\colon f(\mathbf{x}) - \varepsilon \geq 0 \text{ for all } \mathbf{x} \in K\}$
- $f_{inf} \geq f_{sos} = \sup\{\varepsilon\colon f(\mathbf{x}) - \varepsilon = \sum_i s_i g_i,\ s_i \in SOS\}$.

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- $f_{sos}^{(t)} = \sup\{\varepsilon\colon f(\mathbf{x}) - \varepsilon = \sum_i s_i g_i,\ s_i \in SOS,\ \deg(s_i g_i) \leq 2t\}$

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Thm. $\lim_{t\to\infty} f_{sos}^{(t)} = f_{sos} = f_{inf}$ provided e.g. K compact.

Read more in...

1. Monique Laurent. Sums of squares, moment matrices and optimization over polynomials. In Emerging applications of algebraic geometry, volume 149 of IMA Vol. Math. Appl., pages 157–270. Springer, New York, 2009.
2. Murray Marshall. Positive Polynomials and Sums of Squares. American Mathematical Society, 2008.
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How do we get NC polynomials?

Formal **construction:**

1. Start with NC letters $\mathbf{x} := (x_1, \ldots, x_n)$ and operation "multiplication".
2. Consider $\langle \mathbf{x} \rangle$ - monoid freely generated by $\mathbf{x}$ (empty word is 1).
3. Free algebra $\mathbb{R}\langle \mathbf{x} \rangle$: **noncommutative** (nc) polynomials.
4. Add **involution** $*$ which:
   - fixes $\mathbb{R} \cup \{\mathbf{x}\}$ pointwise
   - and reverses words, e.g. $(x_1 x_2^2 x_3 - 2x_3^3)^* = x_3 x_2^2 x_1 - 2x_3^3$.
5. $\operatorname{Sym} \mathbb{R}\langle \mathbf{x} \rangle$ - the set of all **symmetric elements**:

$$\operatorname{Sym} \mathbb{R}\langle \mathbf{x} \rangle = \{ f \in \mathbb{R}\langle \mathbf{x} \rangle \mid f = f^* \}.$$

Why NC polys are relevant?

1. Lots of applications in **control theory**, **systems engineering** and **optimization** (see Helton, McCullough, Oliveira, Putinar, 2008),
2. Applications to **quantum physics** (Pironio, Navascués, Acín, 2010)
3. Applications in **quantum information science** (Pál and T. Vértesi, 2009),
4. **Quantum chemistry** (e.g. to compute the ground-state electronic energy of atoms or molecules) - see cf. Mazziotti, 2004.
5. Certificates of positivity via sums of squares are used to get general bounds on **quantum correlations** (cf. Glauber, 1963).
6. The Bessis-Moussa-Villani conjecture (BMV) from **quantum statistical mechanics** is tackled by NC polynomials (Klep, Schweighofer, 2009; Cafuta, Klep, Povh, 2011)

The BMV conjecture

1. **Bessis - Moussa - Villani (BMV) conjecture (1975):**
   - For symmetric matrices A, B with B positive semidefinite, the function
   $$\Phi^{A,B} : \mathbb{R} \to \mathbb{R}, t \mapsto \mathrm{tr}(e^{A-tB})$$
   is the Laplace transform of a positive measure $\mu^{A,B}$ on $[0, \infty)$:
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2. The conjecture was recently proved by H.R. Stahl: Proof of the BMV conjecture, 2011.

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1. $$\boxed{f(\mathbf{x}) \ \geq \ 0?}$$

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5. We stick to: "Is $f(\mathbf{x}) \;\succeq\; 0$?"

Main questions

Given: real polynomial f in non-commuting variables $\mathbf{x} = (x_1, \ldots, x_n)$.

Q 1: Is

$$f(\underline{A}) \succeq 0$$

for all n-tuples of symmetric matrices $\underline{A} = (A_1, \ldots, A_n)$ of the same size?

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Q 2: Find the smallest **eigenvalue** of f, i.e. compute

$$\boxed{\begin{aligned} \lambda_{\min}(f) \quad = \quad & \inf \langle f(\underline{A})v, v\rangle \\ & \textit{$\underline{A}$ an n-tuple of symmetric matrices,} \\ & \textit{v a unit vector.} \end{aligned}}$$

Sums of hermitian squares (SOHS)

- $SOHS = \{\sum_i g_i^* g_i \colon g_i \in \mathbb{R}\langle \mathbf{x} \rangle\} \subsetneq \operatorname{Sym} \mathbb{R}\langle \mathbf{x} \rangle$.
- $SOHS_d = \{\sum_i g_i^* g_i \colon g_i \in \mathbb{R}\langle \mathbf{x} \rangle,\ \deg(g_i^* g_i) \leq d\} \subsetneq \operatorname{Sym} \mathbb{R}\langle \mathbf{x} \rangle$

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Examples

1. Let $f = x_1x_2 + x_2x_1 + 4 - x_1^2 - x_2^2$. It is not non-negative (take $x_1 = 0,\ x_2 = 3I$).

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Thm.: (Helton, Annals of math., 2002)

$f \in \mathbb{R}\langle \mathbf{x} \rangle$ is SOHS $\Leftrightarrow f(\mathbf{x}) \succeq 0$ whenever we replace x_i by symmetric matrices A_i of dimension $k \times k$, $\forall k \geq 1$.

SDP certificates for SOHS

Problem 1: Given: $f \in \mathbb{R}\langle \mathbf{x} \rangle$, is $f \in SOHS$?

If YES: Provide (**SDP**) certificate!

If NO: Provide (**SDP**) certificate!

SDP certificate for SOHS

Prop.: Suppose $f \in \mathbb{R}\langle \mathbf{x} \rangle$ is of degree $\leq 2d$. Then $f \in SOHS$ if and only if there exists a positive semidefinite (PSD) matrix G satisfying

$$f = W_d^* G W_d,$$

where

$$W_d = \{p \in \langle \mathbf{x} \rangle : \deg(p) \leq d\}.$$

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Rem.: Given such a PSD matrix G with rank r, the SOHS decomposition is

$$f = \sum_{i=1}^{r} g_i^* g_i, \tag{1}$$

where $g_i = H(i,:)W_d$, $G = H^T H$.

SDP certificate for SOHS

Prop.: Given $f = \sum_{w \in \langle \mathbf{x} \rangle} a_w w$ of degree $2d$, then $f \in \mathit{SOHS}$ **iff** exists $G \succeq 0$ such that:

$$\sum_{\substack{p,q \in W_d \\ p^* q = w}} G_{p,q} = a_w, \ \forall w \in W_{2d}$$

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SDP:

$$\begin{array}{llrcll} (\mathrm{SDP}_{\mathrm{SOHS}}) & \inf & \langle I, G \rangle & & & \\ & s.\,t. & \langle A_w, G \rangle & = & a_w + a_{w^*} & \forall w \in W_{2d} \\ & & G & \succeq & 0. & \end{array}$$

where

$$(A_w)_{u,v} = \begin{cases} 2; & \text{if } \ u^* v \in \{w, w^*\}, \ w^* = w, \\ 1; & \text{if } \ u^* v \in \{w, w^*\}, \ w^* \neq w, \\ 0; & \text{otherwise.} \end{cases}$$

Optimization of NC polys.

Problem: Given $f \in \operatorname{Sym}\mathbb{R}\langle \mathbf{x} \rangle$

find smallest eigenvalue of f:

$$\lambda_{\min}(f) \quad = \quad \inf \langle f(A)v, v \rangle$$

A an n-tuple of symmetric matrices,
v a unit vector.

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By Helton-McCullough SOHS theorem:

$$\begin{aligned} \lambda_{\min}(f) &= \sup \quad \lambda \\ \text{s. t.} & \quad f - \lambda \in SOHS_d. \end{aligned} \qquad (\mathrm{SDP}_{\mathrm{eig-min}})$$

Optimization of NC polys.-dual

Prop: The dual to $(\mathrm{SDP}_{\mathrm{eig-min}})$ is

$$\begin{array}{rcll} L_{\mathrm{sohs}} & = & \inf \langle G_f, H \rangle & \\ \text{s. t.} & & H \in \mathcal{S}^+ & \\ & & H_{1,1} = 1 & \\ & & H_{p,q} = H_{r,s} & \text{for all } p,q,r,s, \quad p^*q = r^*s. \end{array} \qquad (\mathrm{DSDP}_{\mathrm{eig-min}})_d$$

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Prop.:

$$L_{\mathrm{sohs}} = \lambda_{\min}(f).$$

Extracting the optimizers

Thm.: (Cafuta, Klep, P., 2010) Let $f \in \mathbb{R}\langle \mathbf{x} \rangle_{\leq 2d}$.

(a) Then $\lambda_{\mathsf{min}}(f)$ is attained if and only if there is a feasible point L for $(\mathrm{DSDP}_{\mathrm{eig-min}})_{d+1}$ satisfying $L(f) = L_{\mathrm{sohs}} = \lambda_{\mathsf{min}}(f)$.

(b) If L_{sohs} is attained, we can find symmetric $s \times s$ matrices $A_1, \ldots, A_n$ and unit vector v such that

$$\lambda_{\mathsf{min}}(f) = \langle f(A)v, v \rangle.$$

Proof:

- Gelfand-Naimark-Segal (GNS) construction
- use flat extensions

Example

```
>> f = (1 - 3*x*y + y*x)'*(1 - 3*x*y + y*x) +
       (-1 + x^2)^2 + (-y+y^2)^2;
>> NCmin(f)
```

$\lambda_{\min}(f) = 0.$

```
>> [X,fX,eig_val,eig_vec]=NCopt(f)
```

Example - cnt.

$$A = \begin{bmatrix} 0.9644 & -0.0379 & -0.1276 & 0.0879 \\ -0.0379 & -0.9828 & 0.1588 & 0.0235 \\ -0.1276 & 0.1588 & 0.4923 & 0.2253 \\ 0.0879 & 0.0235 & 0.2253 & -0.9790 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.8367 & 0.1790 & 0.3326 & 0.0832 \\ 0.1790 & 0.0215 & 0.1388 & 0.5320 \\ 0.3326 & 0.1388 & -0.0227 & -0.6871 \\ 0.0832 & 0.5320 & -0.6871 & -0.1778 \end{bmatrix}$$

$$f(A,B) = \begin{bmatrix} 0.7978 & 1.2130 & 0.8094 & 0.6920 \\ 1.2130 & 3.3989 & -2.6498 & -0.0064 \\ 0.8094 & -2.6498 & 10.5185 & 3.0781 \\ 0.6920 & -0.0064 & 3.0781 & 7.9733 \end{bmatrix}$$

$\lambda_{\min}(f) = \lambda_{\min}(f(A,B)) = 0.0000$,
$v = \begin{bmatrix} -0.8741 & 0.4515 & 0.1789 & 0.0072 \end{bmatrix}^t$.

Note: Commutative min. of f is 0.0625 (for $x = 1,\ y = 1/2$).

SDP complexity

1. $f \in SOHS_d$? has **low** complexity (size of matrix is $\mathcal{O}(kd/2)$) (Newton chip method from P., Klep, 2010)

Read more in...

1. Burgdorf, S., Cafuta, K., Klep, I., and Povh, J. (2013). Algorithmic aspects of sums of hermitian squares. Comp. Optim. Appl. 55(1), 137-153.
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