A Stochastic Derivative-Free Optimization Method with Importance Sampling: Theory and Learning to Control

Adel Bibi[1], El Houcine Bergou[1,2], Ozan Sener[3], Bernard Ghanem[1], Peter Richtárik[1,4]

[1] KAUST, Saudi Arabia [2] MaIAGE, INRA, France [3] Intel Labs [4] MIPT, Russia

We consider the following minimization problem:

$$\min_{x\in\mathbb{R}^n} f(x)$$

Assumption 1 *The objective function f has coordinatewise Lipschitz gradient, with Lispchitz constants $L_1,\dots,L_n>0$. Moreover, f is bounded from below by $f(x_*)\in\mathbb{R}$. That is, f satisfies*

$$f(x_*) \le f(x+te_i) \le f(x) + \nabla_i f(x) t + \frac{L_i}{2}t^2$$

for all $x\in\mathbb{R}^n$ and $t\in\mathbb{R}$, where $\nabla_i f(x)$ is the ith partial derivative of f at x.

Algorithm 1 Stochastic Three Points Method with Importance Sampling ($\mathtt{STP_{IS}}$)

Initialization

Choose initial iterate $x_0\in\mathbb{R}^n$, stepsize parameters $v_1,\dots,v_n>0$ and probabilities $p_1,\dots,p_n>0$ summing up to 1.

For $k=0,1,2,\dots$

1. Select $i_k=i$ with probability $p_i>0$
2. Choose stepsize α_{i_k} proportional to $1/v_{i_k}$
3. Let $x_+ = x_k+\alpha_k e_{i_k}$ and $x_- = x_k - \alpha_k e_{i_k}$
4. $x_{k+1} = \arg\min\{f(x_k), f(x_+), f(x_-)\}$

Assumptions on f	Uniform Sampling Complexity	Importance Sampling	Importance Sampling Complexity [NEW]
None	$\frac{4\sqrt{2}r_0 n^2 L}{\epsilon^2}$	$p_i=\frac{\sqrt{L_i}}{\sum_{i=1}^n\sqrt{L_i}}$	$\frac{4\sqrt{2}r_0\left(\sum_{i=1}^n\sqrt{L_i}\right)^2}{\epsilon^2}$
None	$\frac{4\sqrt{2}r_0 n^2 L}{\epsilon^2}$	$p_i=\frac{L_i}{\sum_{i=1}^n L_i}$	$\frac{4\sqrt{2}r_0 n\left(\sum_{i=1}^n L_i\right)}{\epsilon^2}$
Convex, $R_0<\infty$	$8R_0^2n^2L\left(\frac{1}{\epsilon}-\frac{1}{r_0}\right)$	$p_i=\frac{L_i}{\sum_{i=1}^n L_i}$	$8R_0^2 n\sum_{i=1}^n L_i\left(\frac{1}{\epsilon}-\frac{1}{r_0}\right)$
λ-strongly convex	$\frac{nL}{\lambda}\log\left(\frac{r_0}{\epsilon}\right)$	$p_i=\frac{L_i}{\sum_{i=1}^n L_i}$	$\frac{\sum_{i=1}^n L_i}{\lambda}\log\left(\frac{r_0}{\epsilon}\right)$

Theorem 1 *Let Assumption 1 be satisfied. Choose $\alpha_{i_k}=\frac{\epsilon}{nv_{i_k}}$ where $\sum_{i=1}^n\frac{p_iL_i}{v_i^2}<2n\left(\min_i\frac{p_i}{v_i}\right)$. If*

$$K\ge\frac{2n(f(x_0)-f(x_*))}{\left(\min_i\frac{p_i}{v_i}\right)\left(1-\frac{\sum_{i=1}^n\frac{p_iL_i}{v_i^2}}{2n\left(\min_i\frac{p_i}{v_i}\right)}\right)\epsilon^2},\ \text{then}\ \min_{k=0,1,\dots,K}\mathbb{E}\left[\|\nabla f(x_k)\|_1\right]\le\epsilon.$$

Theorem 2 *Let Assumption 1 be satisfied. Choose $\alpha_{i_k}=\frac{\alpha_0}{v_{i_k}\sqrt{k+1}}$ where $\alpha_0>0$. If*

$$K\ge\frac{2\left(\frac{\sqrt{2}(f(x_0)-f(x_*))}{\alpha_0}+\frac{\alpha_0\sum_{i=1}^n\frac{p_iL_i}{v_i^2}}{2}\right)^2}{\left(\min_i\frac{p_i}{v_i}\right)^2\epsilon^2},\ \text{then}\ \min_{k=0,1,\dots,K}\mathbb{E}\left[\|\nabla f(x_k)\|_1\right]\le\epsilon.$$

Theorem 3 *Let Assumption 1 hold and be satisfied. Choose $\alpha_{i_k}=\frac{|f(x_k+te_{i_k})-f(x_k)|}{tv_{i_k}}$ and sufficiently small t. If*

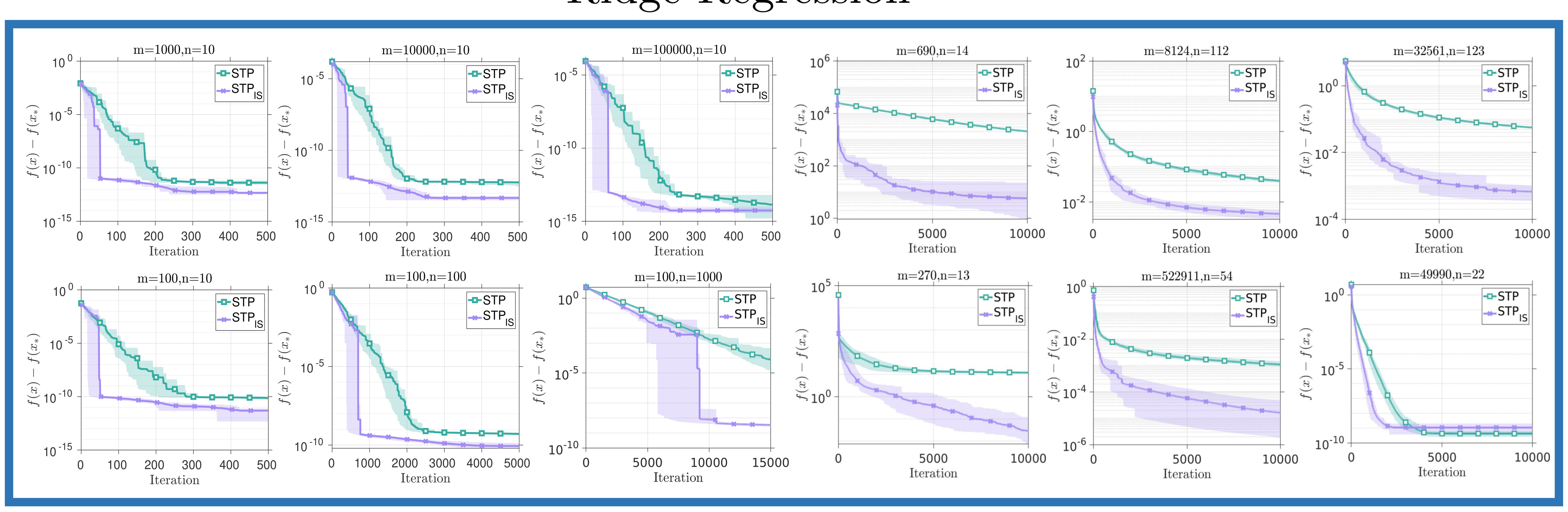
$$k\ge\frac{8R_0^2n}{\min_i\frac{p_i}{v_i}}\left(\frac{1}{\epsilon}-\frac{1}{r_0}\right),\ \text{then}\ \mathbb{E}[f(x_k)]-f(x_*)\le\epsilon.$$

Theorem 4 *Let Assumption 1 hold for λ-strongly convex f. Choose $\alpha_{i_k}=\frac{|f(x_k+te_{i_k})-f(x_k)|}{tv_{i_k}}$ and a sufficiently small t. If*

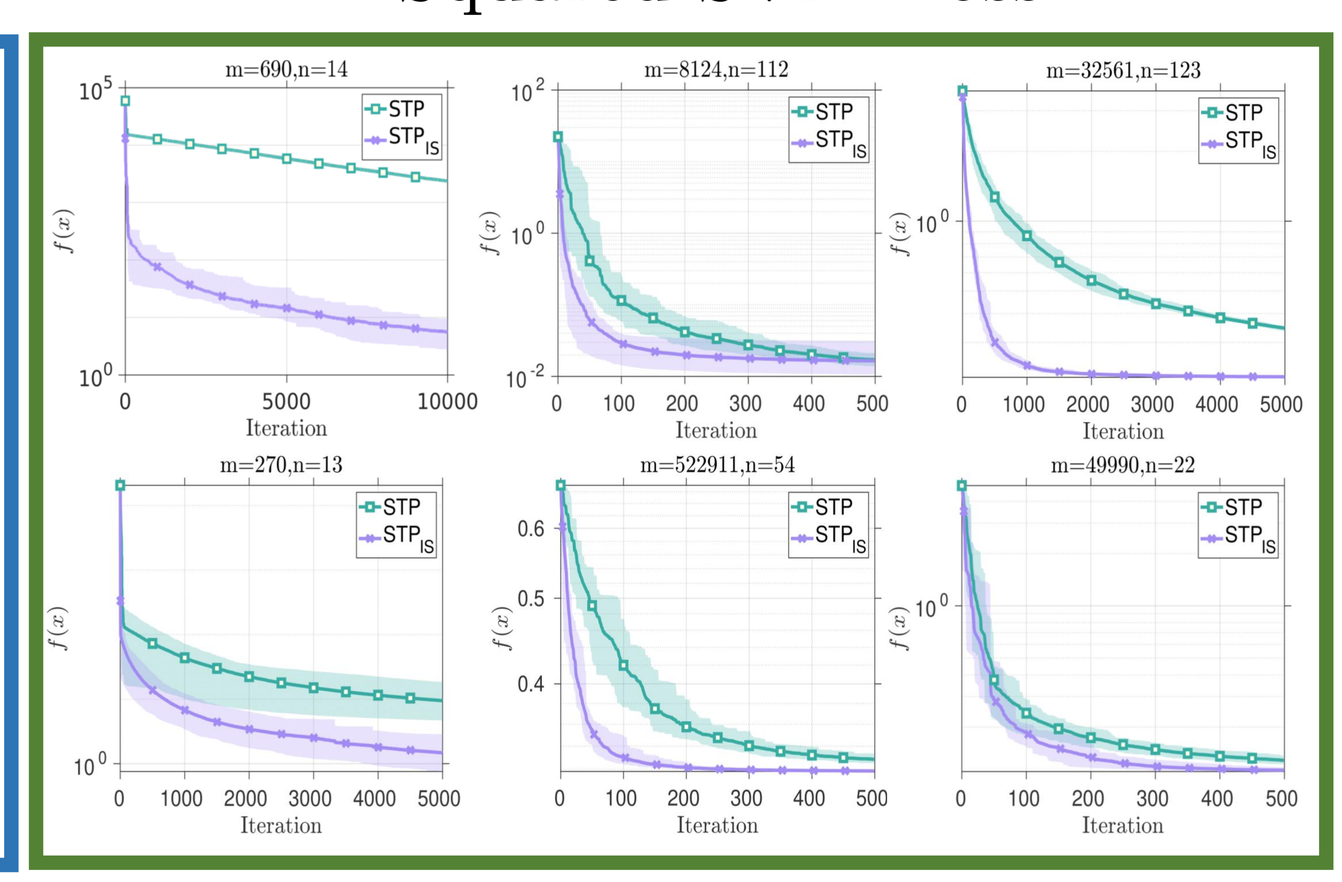
$$k\ge\frac{\max_i\frac{v_i}{p_i}}{\lambda}\log\left(\frac{2\left(f(x_0)-f(x_*)\right)}{\epsilon}\right),\ \text{then}\ \mathbb{E}[f(x_k)]-f(x_*)\le\epsilon.$$

$r_0=f(x_0)-f(x_*)$ $\qquad$ $R_0\stackrel{\text{def}}{=}\max_x\{\|x-x_*\|_\infty\ :f(x)\le f(x_0)\}<\infty$

Ridge Regression

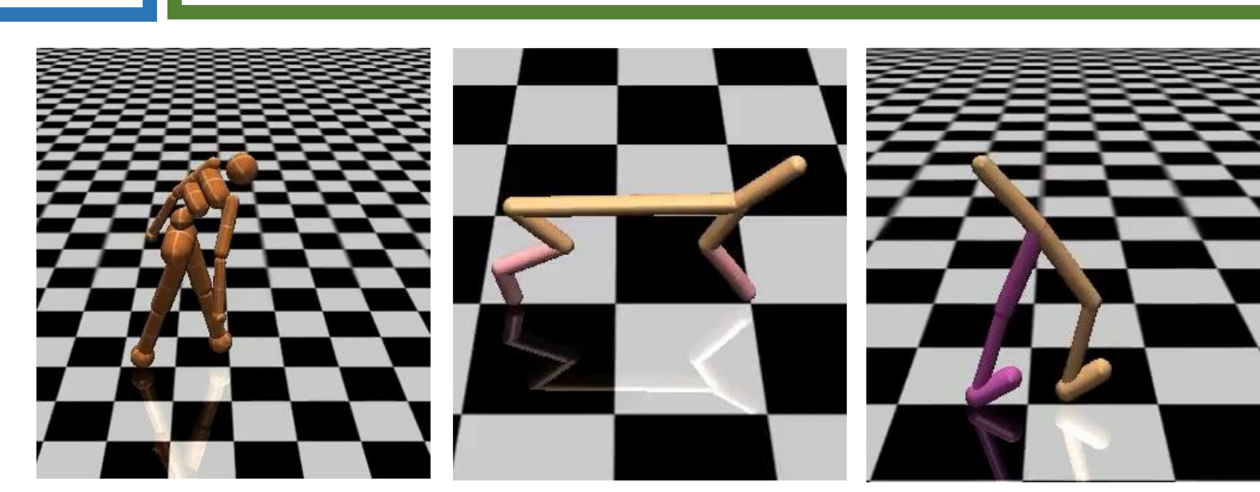


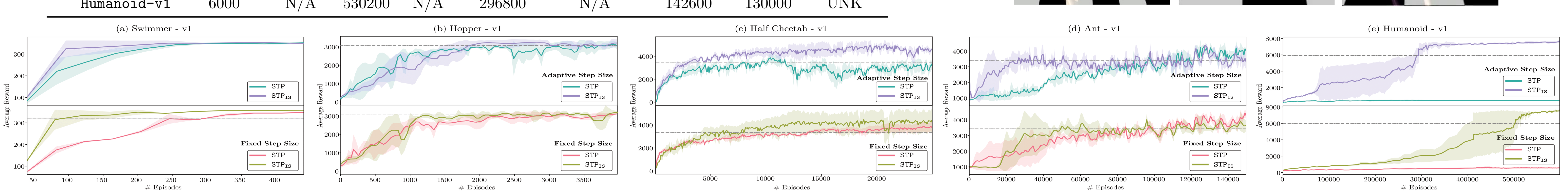
Squared SVM Loss



Continuous Control Experiments:

	Threshold	Fixed Step Size STP	Fixed Step Size $\mathtt{STP_{IS}}$	Adaptive Step Size STP	Adaptive Step Size $\mathtt{STP_{IS}}$	ARS(V1-t)	ARS(V2-t)	NG-lin	TRPO-nn
Swimmer-v1	325	320	110	200	90	100	427	1450	N/A
Hopper-v1	3120	3970	2400	3720	1870	51840	1973	13920	10000
HalfCheetah-v1	3430	13760	4420	5040	2710	8106	1707	11250	4250
Ant-v1	3580	107220	43860	96980	26480	58133	20800	39240	73500
Humanoid-v1	6000	N/A	530200	N/A	296800	N/A	142600	130000	UNK





(a) Swimmer - v1 (b) Hopper - v1 (c) Half Cheetah - v1 (d) Ant - v1 (e) Humanoid - v1

References. "Stochastic Three Points Method for Unconstrained Smooth Minimization". El Houcine Bergou, Eduard Gorbunov, Peter Richtárik.

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