Stochastic Decoupling Method

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Plan

1. Problem structure
2. Examples
3. Proposed method
4. Convergence rates
5. Experiments

Plan

1. Problem structure

Structure

$$\min_x f(x) + \sum_{j=1}^{m} g_j(x)$$

Convex

Structure

$$\min_x f(x) + \sum_{j=1}^{m} g_j(x)$$

Differentiable and smooth

Structure

$$\min_x f(x) + \sum_{j=1}^{m} g_j(x)$$

Proximable

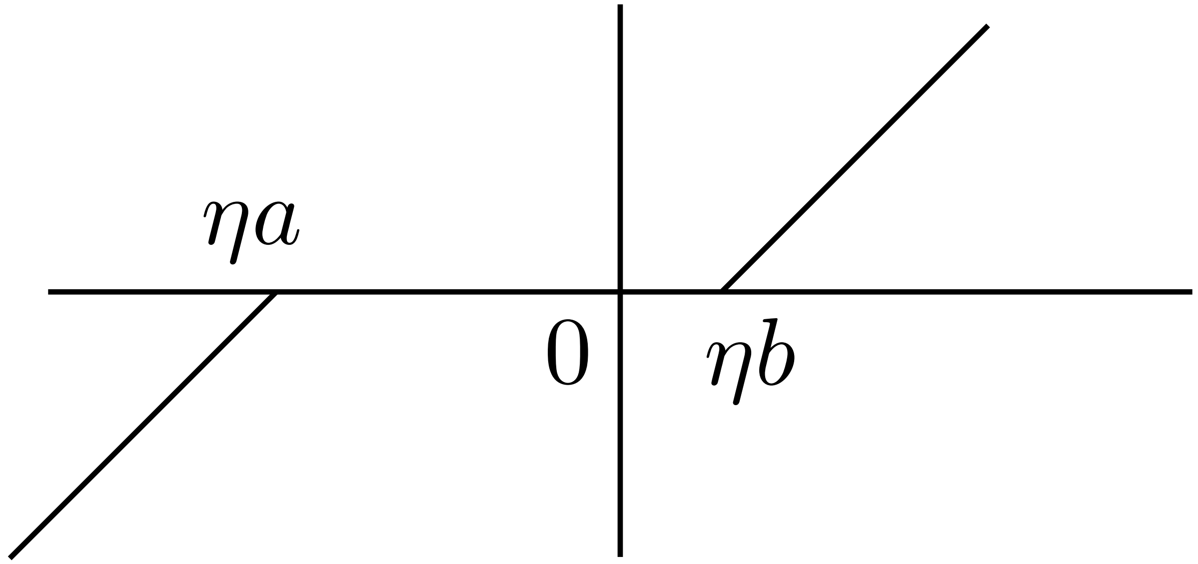
$$\mathrm{prox}_{\eta g}(x) \overset{def}{=} \arg\min_u \left\{ g(u) + \frac{1}{2\eta}\|u - x\|^2 \right\}$$

Structure

Proximable

$$g_j = \begin{cases} ax, & x < 0, \\ bx, & x \geq 0 \end{cases}$$

$$\mathrm{prox}_{\eta g_j}(x) = \begin{cases} x - \eta a, & x < \eta a \\ 0, & \eta a \leq x \leq \eta b \\ x - \eta b, & \text{otherwise} \end{cases}$$



Plan

2. Examples

Examples

$$\min_x f(x) + \sum_{j=1}^{m} g_j(x)$$

$$\min_x f(x) + \|\mathbf{B}x\|_1$$

Examples

$$\min_x f(x) + \sum_{j=1}^m g_j(x)$$

$$\min_x f(x) \text{ s.t. } x \in \bigcap_{j=1}^m C_j$$

Examples

$$\min_{x} f(x) + \sum_{j=1}^{m} g_j(x)$$

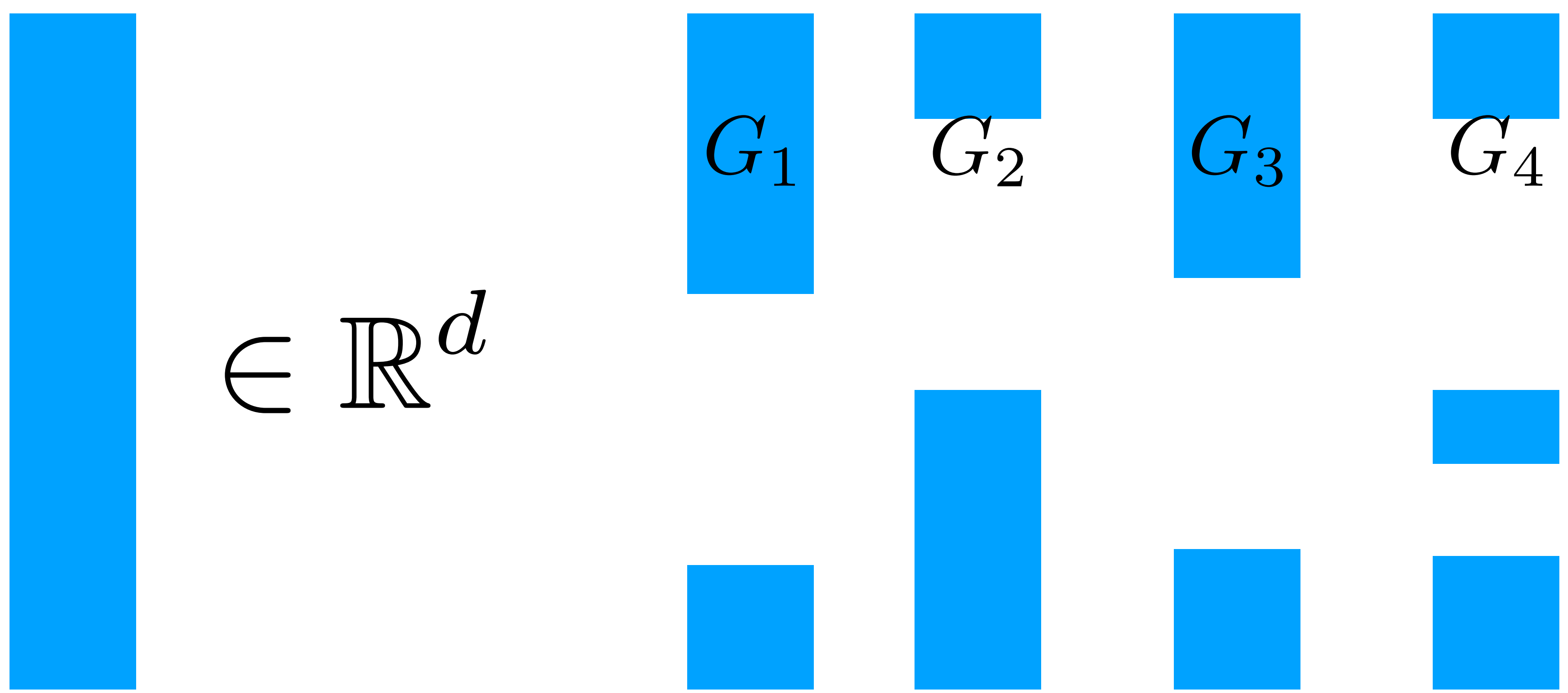
$$\min_{x} \{f(x) \mid \mathbf{C}x = d\}$$

$$\min_{x} \left\{ \frac{1}{n} \sum_{i=1}^{n} f_i(x_i) \mid x_1 = x_2 = \cdots = x_n \right\}$$

Examples

$$\min_x f(x) + \sum_{j=1}^{m} g_j(x)$$

$$\min_x \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \sum_{j=1}^{m} \|x\|_{G_j}$$



Examples

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$$

$$\min_x \frac{1}{2} x^\top \mathbf{A} x + b^\top x$$

$$\min_x \frac{1}{n}\sum_{i=1}^{n} \log\left(1 + \exp(-b_i a_i^\top x)\right)$$

$$\min_x \frac{1}{n}\sum_{i=1}^{n} l\left(b_i, \Phi(x, a_i)\right) \qquad a_i \in \mathbb{R}^{d_1}, b_i \in \mathbb{R}$$

Examples

$$\min_x f(x) + \sum_{j=1}^{m} g_j(x)$$

$$\frac{1}{2}\|y - \mathbf{A}x\|_2^2 + \lambda\|x\|_1 + \lambda_1 \sum_{j=1}^{m} \|\mathbf{R}_j x\|_2$$

“We have not experimented with this yet, as the computation seems challenging due to the presence of ℓ_2 norms.”
(Tay, Friedman, Tibshirani, PCA-Lasso 2018)

Plan

3. Proposed method

Gradient descent

$$x^{t+1} = x^t - \eta \nabla f(x^t), \quad t = 0, 1 \ldots, T$$

Proximal gradient descent

$$x^{t+1} = \mathrm{prox}_{\eta g}(x^t - \eta \nabla f(x^t))$$

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Gradient descent

$$x^{t+1} = \mathrm{prox}_{\eta g}(x^t - \eta \nabla f(x^t))$$

$$f(x^t) - \min_x f(x) = \mathcal{O}\left(\left(1 - \frac{\mu}{L}\right)^t\right)$$

Linear rate

Stochastic decoupling

$$y^t = \frac{1}{m}\sum_{j=1}^{m} y_j^t$$

$$z^t = x^t - \eta \nabla f(x^t) - \eta y^t$$

Stochastic decoupling

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$$z^t = x^t - \eta \nabla f(x^t) - \eta y^t$$

$$y_j^t \approx \partial g_j(x^t), \quad y^t \approx \partial g(x^t)$$

$$z^t \approx \mathrm{prox}_{\eta g}(x^t - \eta \nabla f(x^t))$$

Stochastic decoupling

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$$z^t \approx \mathrm{prox}_{\eta g}(x^t - \eta\nabla f(x^t))$$

$$x^{t+1} = \mathrm{prox}_{\eta g_j}(z^t + \eta y_j^t)$$

$$y_j^{t+1} = y_j^t + \frac{1}{\eta}(z^t - x^{t+1}) \in \partial g_j(x^{t+1})$$

Plan

4. Convergence rates

Convergence

$\mathcal{O}(1/\varepsilon)$ **Convex**

Convergence

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$\mathcal{O}(1/\sqrt{\varepsilon})$ f is μ-strongly convex

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$\mathcal{O}(1/\sqrt{\varepsilon})$ f is μ-strongly convex

$\mathcal{O}(\log \frac{1}{\varepsilon})$ $g_j(x) = \phi_j(a_j^\top x)$

$\mathbf{A}^\top \mathbf{A} \succ 0$

Convergence

$\mathcal{O}(1/\varepsilon)$ Convex

$\mathcal{O}(1/\sqrt{\varepsilon})$ f is μ-strongly convex

$\mathcal{O}(\log \frac{1}{\varepsilon})$ $g_j(x) = \phi_j(a_j^\top x)$

$\mathbf{A}^\top \mathbf{A} \succ 0$

Was only possible for $f = \frac{1}{2}\|x - x^0\|^2$

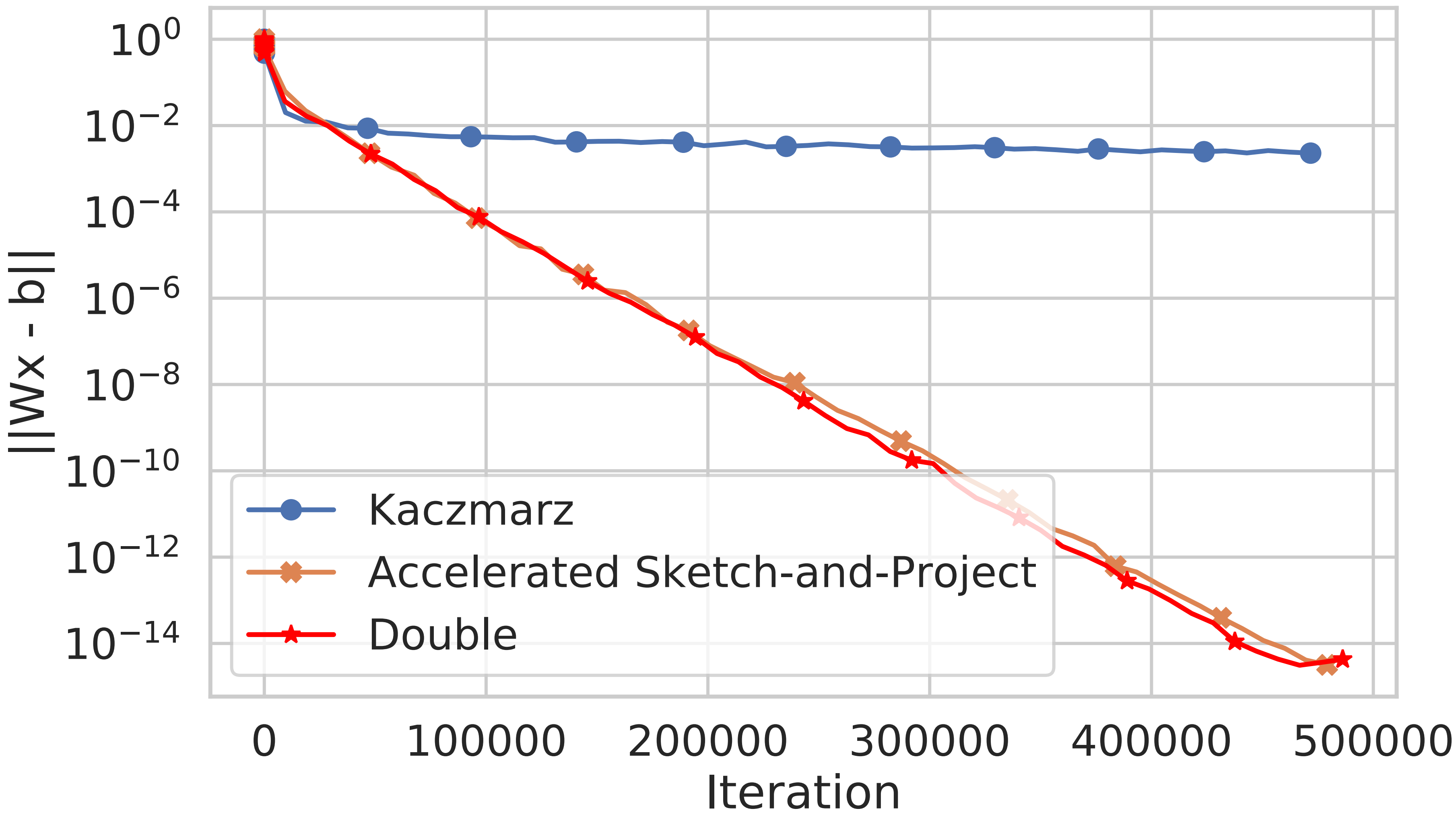
before our work

Plan

5. Experiments

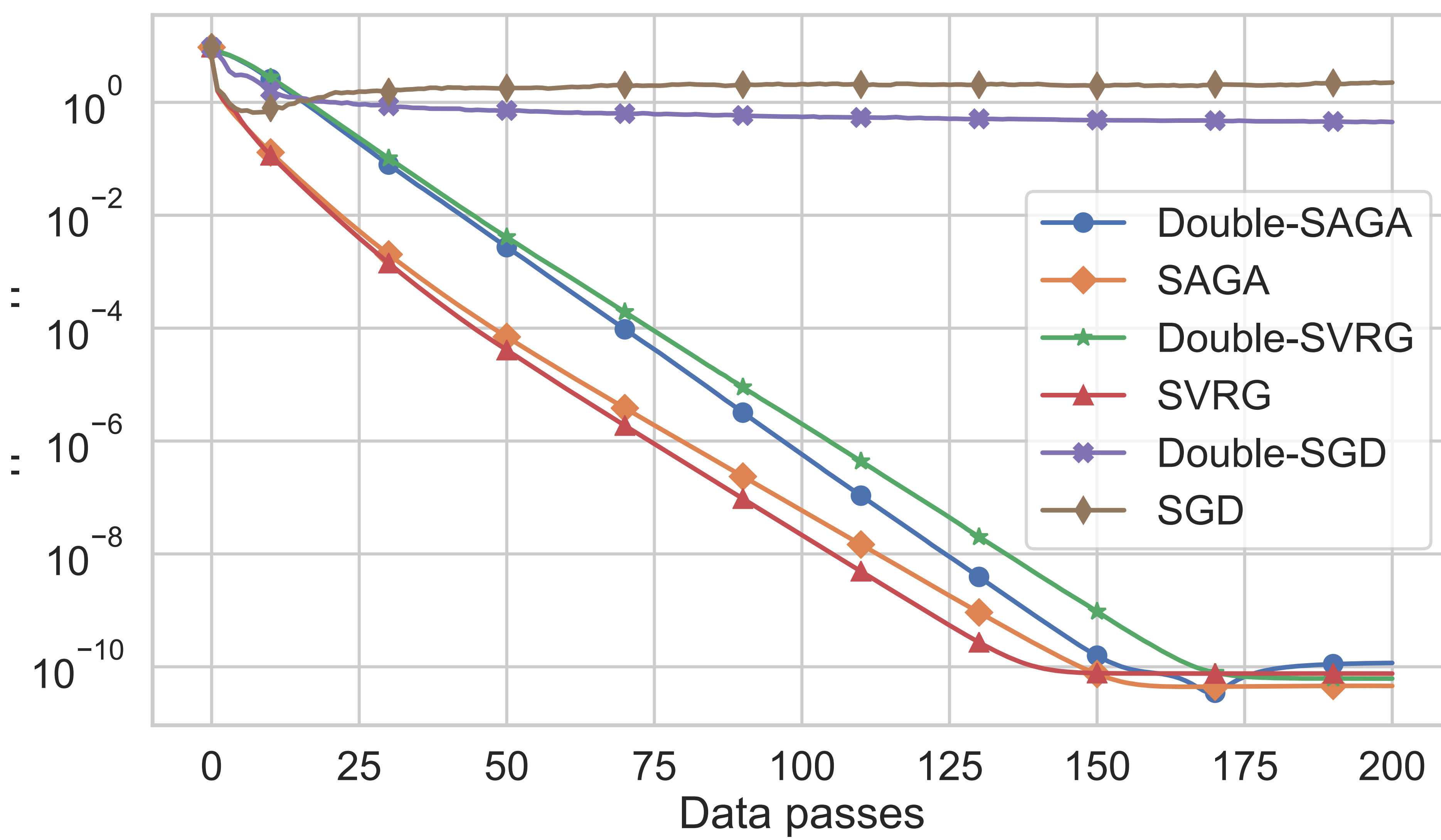
Experiments

$$\min_x \{ \|x - x^0\| \mid \mathbf{W}x = b \}$$



Experiments

$$\min_x \left\{ \frac{1}{2} x^\top \mathbf{A} x + b^\top x \mid \mathbf{C}x = d \right\}$$



Reference

A Stochastic Decoupling Method for Minimizing the Sum of Smooth and Non-Smooth Function

arXiv:1905.11535